Stochastic modeling of mesoscale eddies in barotropic wind-driven circulation

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Abstract

We propose to follow a recent stochastic quasi–geostrophic model [4] derived from a decomposition of the flow into a resolved component and a time–uncorrelated uncertainty [3]. One important characteristic of this random model is that it conserves the total energy along each realization. Such a stochastic principle has been tested for the numerical simulation of the wind-driven circulation in a shallow ocean basin [1]. The numerical assement shows that the proposed random model can well capture on a coarse mesh the correct four–gyre time–averaged circulation structure, as predicted by a direct numerical simulation at a much finer resolution.

Model

1. Decomposition of Eddy–Mean Flow.

$$\frac{\mathrm{d}\boldsymbol{X}_t}{\mathrm{d}t} = \boldsymbol{u} + \dot{\boldsymbol{\eta}}_t, \qquad \dot{\boldsymbol{\eta}}_t(\boldsymbol{x}) = \sum_k^{\infty} \sqrt{\lambda_k}\boldsymbol{\Phi}_k(\boldsymbol{x})\xi_k \quad (1)$$

where the mesoscale eddy is represented by a time–uncorrelated noise with $\xi_k \sim \mathcal{N}(0,1)$ i.i.d., yet correlated in space with its covariance operator defined as:

$$\boldsymbol{Q}_t(\boldsymbol{x},\boldsymbol{y}) = \sum_k \lambda_k \boldsymbol{\Phi}_k(\boldsymbol{x})\boldsymbol{\Phi}_k^T(\boldsymbol{y}) \quad (2)$$

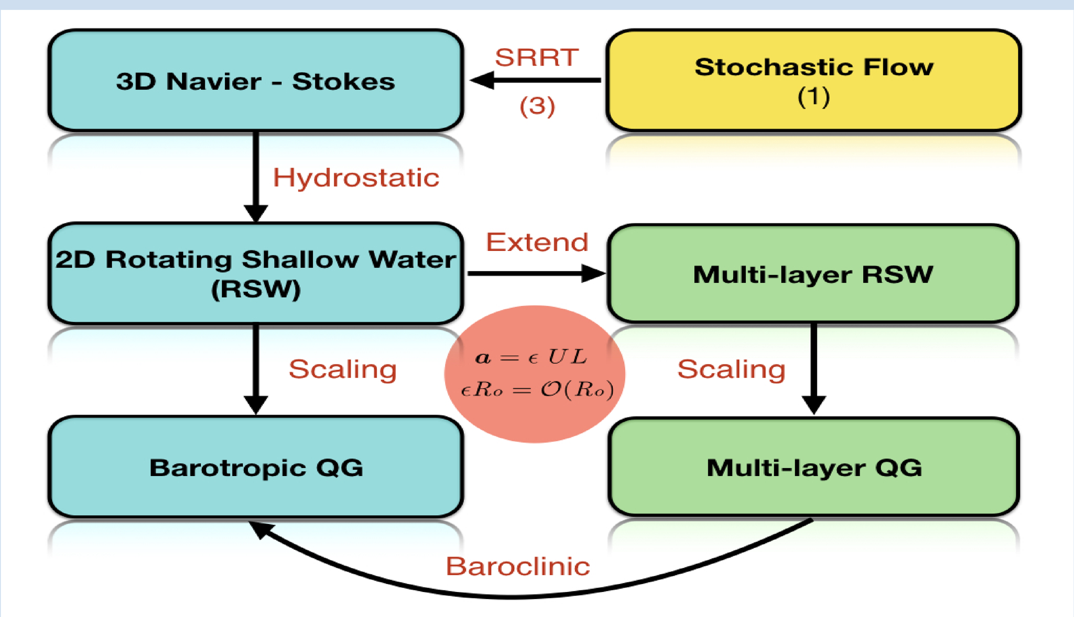
2. Stochastic Reynolds Transport Theorem (SRRT). Following the framework of [3], the rate of change of a scalar process θ within a volume, transported by the random flow (1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} \theta \mathrm{d}V = \int_{\mathcal{V}(t)} \Big(\frac{\mathrm{D}\theta}{\mathrm{D}t} + \theta\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{u} - \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{a})\Big)\mathrm{d}V \quad (3)$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} \triangleq \frac{\partial\theta}{\partial t} + (\boldsymbol{u} + \dot{\boldsymbol{\eta}}_t)\boldsymbol{\cdot}\boldsymbol{\nabla}\theta - \sum_{i,j=1}^{3} \partial_{ij}^2(a_{ij}\theta) \quad (4)$$

where $\boldsymbol{a}(\boldsymbol{x},t) \triangleq \frac{\mathrm{d}t}{2}\boldsymbol{Q}_t(\boldsymbol{x},\boldsymbol{x})$ is defined as a one–point one–time variance tensor and the eddies are assumed to be incompressible, i.e. $\boldsymbol{\nabla}\boldsymbol{\cdot}\dot{\boldsymbol{\eta}}_t = 0$.

3. Derivation of QG Model.



4. Stochastic Barotropic Vorticity Equation (SBVE). The governing equations for SBVE can be written in dimensionless form as:

$$\frac{\partial q}{\partial t} + \mathrm{J}(\psi, q) = F + D + M \quad (5)$$

$$M = \boldsymbol{e}_3\boldsymbol{\cdot}\boldsymbol{\nabla}\times\Big((\dot{\boldsymbol{\eta}}_t\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{u} - \sum_{i,j=1}^{2}\partial_{ij}^2(a_{ij}\boldsymbol{u})\Big) + \boldsymbol{e}_2\boldsymbol{\cdot}\dot{\boldsymbol{\eta}}_t \quad (6)$$

where $q = Ro\nabla^2\psi + y$ is the potential vorticity in the beta plane with Ro the Rossby number, ψ is the mean velocity streamfunction, J is the Jacobian operator, $F = \sin(\pi y)$ is the symmetric double–gyre forcing, and D is an entrophy–dissipation operator which can be given by $(\frac{\delta_2}{L})^3\nabla^4\psi$ or $(\frac{\delta_4}{L})^5\nabla^6\psi$, with δ_2, δ_4 the width of the Munk boundary layer. More details on the physical mechanism is referred to [1]. We denote that, without any forcing and dissipation, SBVE conserves the mean kinetic energy along each realization:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\frac{1}{2}\|\boldsymbol{\nabla}\psi\|^2\mathrm{d}\boldsymbol{x} = 0. \quad (7)$$

References

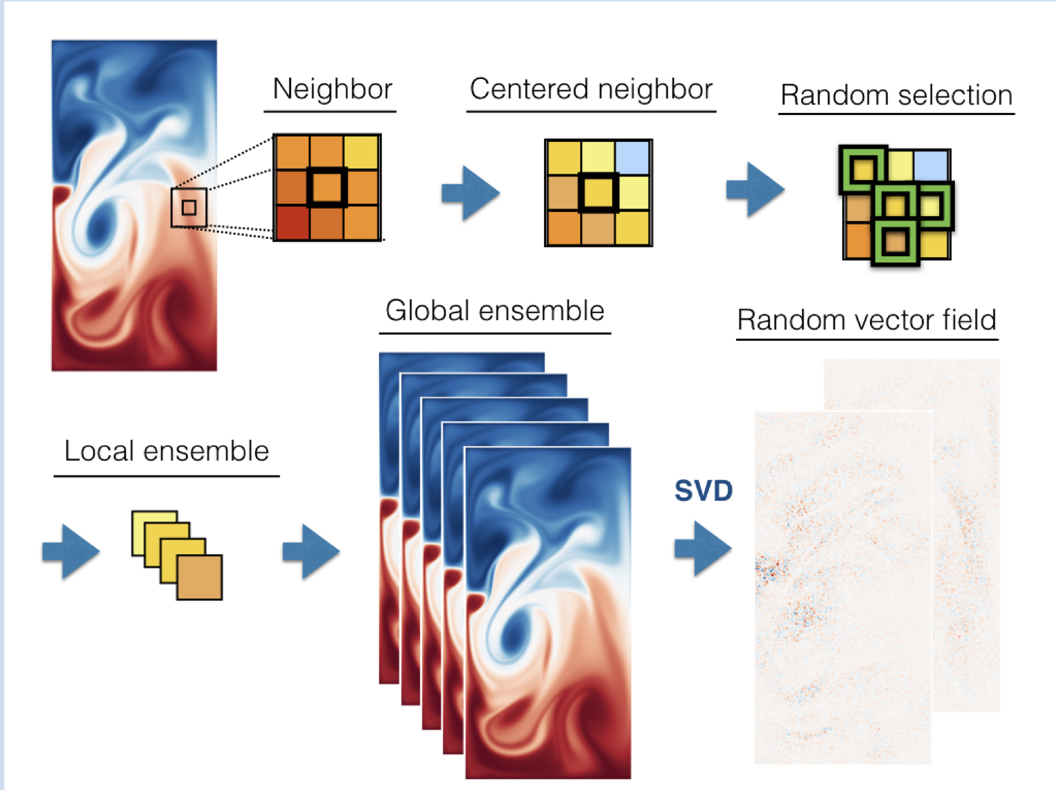
[1] D.D. Holm, B.T. Nadiga: *Modeling mesoscale turbulence in the barotropic double–gyre circulation*, Journal of Physical Oceanography, 2355–2365 (2003)

[2] S. Kadri Harouna, E. Mémin: *Stochastic representation of the Reynolds transport theorem: revisiting large–scale modeling*, Computers and Fluids, 456–469 (2017)

[3] E. Mémin: *Fluid flow dynamics under location uncertainty*, Geophysical & Astrophysical Fluid Dynamics, 119–146 (2014)

[4] V. Resseguier, E. Mémin, B. Chapron: *Geophysical flows under location uncertainty*, Geophysical & Astrophysical Fluid Dynamics, 177–208 (2017)

On–line learning of eddy's EOFs from mean flow



Method: 1) Sliding window to draw N_o pseudo–observations at each grid point, removing average to substract fluctuations within each window size (L): $\boldsymbol{u}'_L = \boldsymbol{u}_L - \langle\boldsymbol{u}_L\rangle$;
2) Apply singular value decomposition (SVD) of fluctuations to get a set of empirical orthogonal functions (EOFs): $\dot{\boldsymbol{\eta}}_t^{(L)} = \sum_{k=1}^{N_o-1}\sqrt{\lambda_k}\boldsymbol{\Phi}_k^{(L)}\xi_k$;
3) Adapt variance to grid scale (l) using a turbulence power–law [2]: $\dot{\boldsymbol{\eta}}_t^{(l)} = (\frac{l}{L})^{1/3}\dot{\boldsymbol{\eta}}_t^{(L)}$.

Benefits: parameter–free, no need of data, time–dependent EOFs, scaling auto–adapted.

Time–statistical nature of double–gyre circulation

Model	Grid spacing	Munk scale	Ro	Boundary condition	Running time
DNS	0.004	$\delta_2 = 0.02$	0.04^2	free–slip	140
LES	0.05	$\delta_4 = 0.15$	0.04^2	free–slip	140
SBVE	0.05	$\delta_4 = 0.15$	0.04^2	free–slip	140

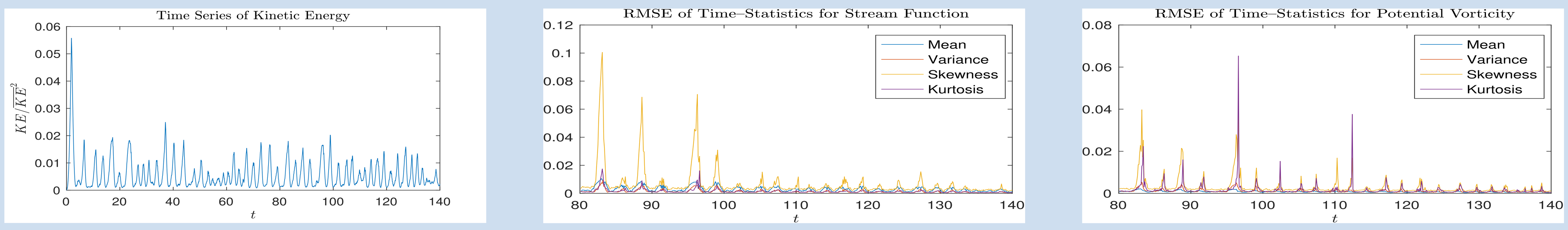


Figure 1: Test of convergences for the stationary states – From left to right: time serie of kinetic energy; RMSEs of statistics for streamfunction by perturbation of time intervals; RMSEs of statistics for potential vorticity.

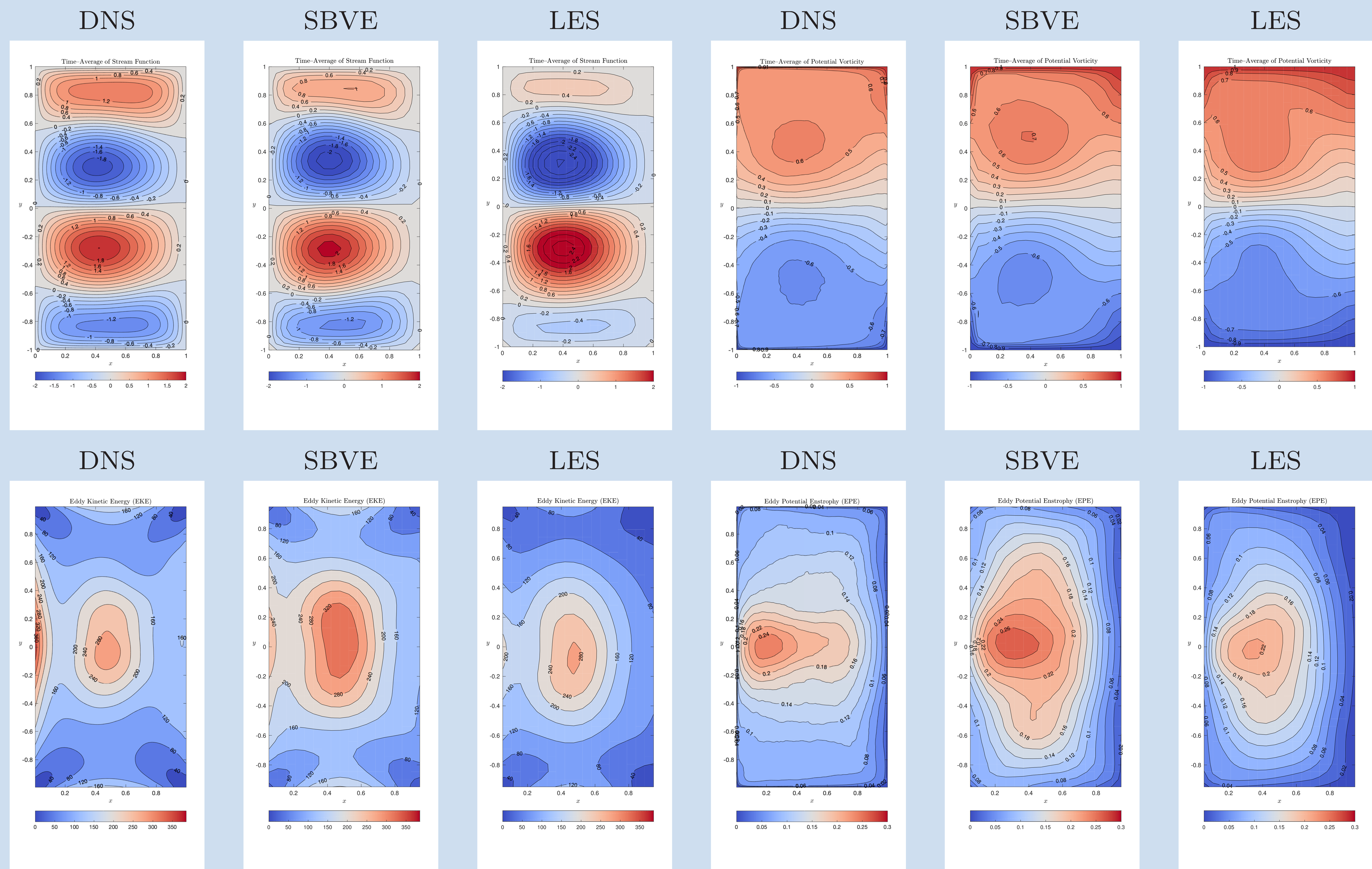


Figure 2: Comparison of contour plots for different statistics: left top three – time–averaged streamfunctions; right top three – time–averaged potential vorticity; left bottom three – eddy kinetic energy (EKE); right top three – eddy potential enstrophy (EPE).

Conclusion

The proposed random model has an explicit eddy representation which respects a set of physical consistency laws, such as SRRT (3) and energy conservation (7). From the numerical results of the time–statistics, we understand that the random model helps us to correct the wrong numerical dissipation of energy due to the Laplacian/Hyper–viscosity. It results that the physical four–gyre pattern in time average are recovered on a coarse–grid computation. In another work, the performance of our random model has been evaluated and analyzed in terms of uncertainty quantification and ensemble forecasting. Therein, results confirm that the spread is more accurate compared to a deterministic model with a pertubation of the initial condition. This ability is in particular essential for data assimilation applications. For future work, we attempt to move on to a multilayer model including baroclinicity. The objective will be to understand more about the isopycnal mixing through a specific parametrization.